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Radial position–momentum uncertainties for the Dirac hydrogen-like atoms

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Abstract

We show how the radial position–momentum uncertainty product can be obtained analytically for the Dirac hydrogen-like atoms. Some interesting features for this system are found. First, for the same principal quantum number n, as the azimuthal quantum number l increases, the uncertainties Δp_r and $\Delta r \Delta p_r$ decrease. However, the uncertainty Δr does not always decrease, which is different from the non-relativistic hydrogen-like atoms case, where it always decreases. Second, for the same l both Δr and $\Delta r \Delta p_r$ increase as n increases. Third, all uncertainties for the same n and $l = n - 1$ are smallest in comparison with those for the same n but $l \neq n - 1$. Fourth, the uncertainty $\Delta r \Delta p_r$ is independent of the value of charge Z in the non-relativistic case, while it is related with the value of charge Z in the relativistic case. Fifth, the relativistic corrections to the non-relativistic values of uncertainties are very small when the values of charge Z are not too big, while the relativistic corrections to them will appear explicitly for a large value of charge Z.

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1. Introduction

The great physicist Heisenberg introduced into physics a famous 'uncertainty principle' which is one of the deepest philosophical contributions in quantum mechanics. The frequently quoted statement of this principle addresses: 'It is impossible to know both the position and the momentum of a particle at a given momentum to an arbitrary degree of accuracy' [1], which is expressed as $\Delta x \Delta p \geqslant \hbar/2$.

It is usually known that the position–momentum uncertainty products for the harmonic oscillator and non-relativistic hydrogen atom were obtained analytically 75 years ago [2].

Some decades later, the position–momentum uncertainty relations for several exactly solvable potentials such as the symmetric Pöschl–Teller, symmetric Rosen–Morse and one-dimensional Morse potentials were carried out and discussed [3]. It should be noted that almost all contributions to this topic are limited to the one-dimensional Schrödinger equation. Nevertheless, the radial position–momentum uncertainties for the non-relativistic hydrogen-like atoms in three dimensions have been studied recently by Kuo [4]. Due to their importance in physics, we attempt to show how the position–momentum uncertainty relations for the Dirac hydrogen-like atoms can be obtained analytically, which is the main purpose of this work.

This paper is organized as follows. In section 2 we present the exact uncertainties Δr, Δp_r and $\Delta r \Delta p_r$ for the Dirac hydrogen-like atoms with the aid of the recently developed MATHEMATICA package INTEPFFLL [5]. Section 3 is devoted to the study of some interesting properties of these uncertainties by taking a few typical values of charge Z. In section 4 we study two special cases for $l = 0$ and $l = n - 1$. The non-relativistic limit is carried out in section 5. Some concluding remarks are given in section 6.

2. Relativistic hydrogen-like atoms

Let us first mention the convention used in writing the exact solutions of the Dirac hydrogen-like atoms before studying the mean values of the position r and radial momentum p_r.

Generally speaking, we can write down the Dirac equation as follows:

$$[c\alpha \cdot p + \beta mc^2 + V(r)]\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{1}$$

where α and β are the Dirac matrices given by

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

If we make use of the convenient form [6–8]

$$\Psi(\mathbf{r}) = \begin{pmatrix} if(r)\Omega_{jlm}(\theta,\varphi) \\ -g(r)\Omega_{jl'm}(\theta,\varphi) \end{pmatrix}, \tag{3}$$

where $l = j \pm 1/2$, $|l - l'| = 1$ and $\Omega_{jlm}(\theta,\varphi)$ are the spherical spinors, then the exact solutions of the Dirac equation with a Coulomb-like potential $V(r) = -Ze/r$ can be expressed as follows ($c = 1$) [6, 9],

$$\left.\begin{matrix} f(r) \\ g(r) \end{matrix}\right\} = \pm\frac{(2\beta)^{3/2}}{\Gamma(2\gamma+1)}\sqrt{\frac{(1\pm\epsilon)\Gamma(2\gamma+n'+1)}{n'!4N(N-\kappa)}}\,\mathrm{e}^{-\beta r}(2\beta r)^{\gamma-1}$$
$$\times [\mp n' F(-n'+1, 1+2\gamma, 2\beta r) + (N-\kappa)F(-n', 1+2\gamma, 2\beta r)], \tag{4}$$

with the following dimensionless parameters

$$\beta = \frac{Z}{Na_0}, \qquad \gamma = \sqrt{\kappa^2 - Z^2\alpha^2}, \qquad N = \sqrt{n^2 - 2n'(|\kappa| - \gamma)},$$
$$a_0 = \frac{mc\alpha}{\hbar}, \qquad \epsilon = \frac{E}{E_0} = \left[1 + \frac{Z^2\alpha^2}{(n'+\gamma)^2}\right]^{-1/2} = \frac{n'+\gamma}{N}, \tag{5}$$
$$n = n' + |\kappa| = n' + j + 1/2, \qquad n' = 0, 1, 2, \ldots,$$

where the fine structure constant is taken as $\alpha \simeq 1/137$. The E and $E_0 = mc^2$ denote the energy and the rest energy of the electron, respectively, and κ is defined by

$$\kappa = \begin{cases} -(j+1/2) = -(l+1), & \text{for } j = l + 1/2, \\ (j+1/2) = l, & \text{for } j = l - 1/2. \end{cases} \tag{6}$$

Now, we derive the position uncertainty Δr for the Dirac hydrogen-like atoms case. To this end, from equation (4) we can evaluate the mean values of r^s as follows:

$$\begin{aligned}\langle r^s\rangle &= \int_0^\infty r^{s+2}(|f(r)|^2+|g(r)|^2)\,\mathrm{d}r\\ &= \frac{2^{-s}\Gamma(1+n'+2\gamma)}{N(N-\kappa)n'!\Gamma(1+2\gamma)^2}[n'^2 I_{FF}(n'-1,0,1+2\gamma,0,s-1)\\ &\quad+2n'\epsilon(-N+\kappa)I_{FF}(n'-1,1,1+2\gamma,0,s-1)\\ &\quad+(N-\kappa)^2 I_{FF}(n',0,1+2\gamma,0,s-1)],\end{aligned}\qquad (7)$$

where we have used our developed MATHEMATICA package INTEPFFLL to compute these complicated integrals including the product of the confluent hypergeometric functions [5, 10, 11]

$$I_{FF}(n,\Delta n,\mu,\Delta\mu,\lambda) = \int_0^\infty \mathrm{e}^{-x}x^{\mu+\lambda}F(-n,\mu;x)F[-(n+\Delta n),\mu+\Delta\mu;x]\,\mathrm{d}x, \qquad (8)$$

where Δn, $\Delta\mu$ and λ are integers and $\Delta n \geqslant 0$.

Here we want to make two useful remarks about equation (8). First, this integral formula for some different choices of Δn, $\Delta\mu$ and λ can be obtained analytically as shown in our recent works [5, 10, 11]. Also, it should be pointed out that our developed package INTEPFFLL are enough to be used to perform all the present calculations of uncertainty relations for the Dirac hydrogen-like atoms even though this program has its own limitation. Second, it is well known that the confluent hypergeometric functions can be written as the form of the associated Laguerre polynomials [15]

$$F(-n,\mu;x) = \frac{n!\Gamma(\mu)}{\Gamma(n+\mu)}L_n^{\mu-1}(x), \qquad n=0,1,2,\ldots, \quad \Re e\mu>0. \qquad (9)$$

Using this, we can define

$$\begin{aligned}I_{LL}(n,m,\mu,\nu,\lambda) &= \int_0^\infty \mathrm{e}^{-x}x^{\lambda}F(-n,\mu;x)F(-m,\nu;x)\,\mathrm{d}x,\\ &= \Gamma(\mu)\Gamma(\nu)\frac{n!m!}{\Gamma(n+\mu)\Gamma(m+\nu)}\int_0^\infty \mathrm{e}^{x}x^{\lambda}L_n^{\mu-1}(x)L_m^{\nu-1}(x)\,\mathrm{d}x,\end{aligned}\qquad (10)$$

where $n, m = 0, 1, 2, \ldots$, and $\Re e\mu > 0$, $\Re e\nu > 0$ and $\Re e\lambda > -1$. Therefore, we notice that this formula is more general than I_{FF} given in equation (8) since μ and ν are not limited to integers only. To show the existence of an exact analytical formula for equation (10), we want to briefly sketch this result following [12–14], from which one can obtain the following result:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-x}x^{\lambda}L_n^{\mu-1}(x)L_m^{\nu-1}(x)\,\mathrm{d}x &= \Gamma(\lambda+1)\frac{\Gamma(n+\mu-\lambda-1)\Gamma(m+\nu-\lambda-1)}{n!m!\Gamma(\mu-\lambda-1)\Gamma(\nu-\lambda-1)}\\ &\quad\times {}_3F_2(-n,-m,\lambda+1;\lambda-\mu-n+2,\lambda-\nu-m+2;1),\end{aligned}\qquad (11)$$

where ${}_3F_2$ is a generalized hypergeometric series. It can be easy to compute by using its well-known definition

$${}_3F_2(-n,-m,a;b,c;1) = \sum_{k=0}^{\min[n,m]}\frac{(-n)_k(-m)_k(a)_k}{(b)_k(c)_k k!}, \qquad (12)$$

where $(x)_k$ stands for the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}. \qquad (13)$$

Thus, we can finally express equation (10) as follows:

$$I_{LL}(n, m, \mu, \nu, \lambda) = \frac{\Gamma(\mu)\Gamma(\nu)\Gamma(\lambda+1)}{\Gamma(n+\mu)\Gamma(m+\nu)} \frac{\Gamma(n+\mu-\lambda-1)\Gamma(m+\nu-\lambda-1)}{\Gamma(\mu-\lambda-1)\Gamma(\nu-\lambda-1)} \times {}_3F_2(-n, -m, \lambda+1; \lambda-\mu-n+2, \lambda-\nu-m+2; 1). \quad (14)$$

It should be pointed out that, in principle, we can make use of these two formulae (8) and (14) to obtain all results that we require. Nevertheless, we can quickly obtain all required calculations of the uncertainty relations for the Dirac hydrogen-like atoms with the aid of our developed MATHEMATICA program. Moreover, it should be noted that the results obtained from this MATHEMATICA package are also analytical, as shown in our recent works and present paper [10, 11]. In particular, such a tedious task can be easily and quickly performed by this MATHEMATICA package [5].

We now go about to study the uncertainty relations for the Dirac hydrogen-like atoms. In order to calculate the uncertainty Δr, we have to know the mean values of r and r^2 for an electron in the Dirac hydrogen-like atoms case. Fortunately, they have been already known in our recent work [11]:

$$\langle r\rangle = \frac{a_0}{2Z}\left[-\kappa + \frac{(n'+\gamma)[3n'(n'+2\gamma)+2\kappa^2]}{N}\right], \quad (15)$$

$$\langle r^2\rangle = \frac{a_0^2}{2Z^2}[n'(n'+2\gamma)(1+5n'^2+10n'\gamma+4\gamma^2) + (1+3n'^2 + 6n'\gamma+2\gamma^2)\kappa^2 - 3(n'+\gamma)\kappa N]. \quad (16)$$

The uncertainty Δr in the measurement of the distance of the electron from the nucleus can be obtained as follows:

$$\begin{aligned}\Delta r &= \sqrt{\langle r^2\rangle - \langle r\rangle^2}\\ &= \frac{a_0}{2ZN^2}[n'^2(n'+2\gamma)^2(2+n'^2+2n'\gamma-\gamma^2) + n'(n'+2\gamma)(3+4n'(n'+2\gamma))\kappa^2\\ &\quad - 2N(n'+\gamma)\kappa^3 + (1+2n'(n'+2\gamma))\kappa^4]^{1/2},\end{aligned} \quad (17)$$

which means that Δr is inversely proportional to the value of charge Z except for its dependence on the principal quantum number n ($n' = n - |\kappa|$) and azimuthal quantum number l. The latter can take n different values in all for a given n with $l = 0, 1, \ldots, n-1$. On the other hand, the relative dispersion $\Delta r/\langle r\rangle$ in the measurement of radial position, which also depends on the quantum numbers n and l as well as the value of charge Z, will give us a possible estimate of how indeterminate the measurement in radial position r is relative to the mean value $\langle r\rangle$ of the radial position r for an electron in the Dirac hydrogen-like atoms.

In order to obtain the product $\Delta r \Delta p_r$ of the uncertainties Δr and Δp_r, we have to study the radial momentum uncertainty Δp_r. To this end, we begin by considering the definition of the radial momentum

$$p_r = -\mathrm{i}\hbar \frac{1}{r}\frac{\partial}{\partial r} r = -\mathrm{i}\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \quad (18)$$

from which we can obtain $\langle p_r\rangle = 0$. This coincides with the result for other exactly solvable potentials [2–4].

Now, let us derive the mean value of p_r^2

$$\begin{aligned}\langle p_r^2\rangle &= (-\mathrm{i}\hbar)^2\int_0^\infty r^2\{f(r), g(r)\}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\{f(r), g(r)\}^T)\,\mathrm{d}r\\ &= \hbar^2\int_0^\infty r^2\left[\left(\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}g(r)}{\mathrm{d}r}\right)^2\right]\mathrm{d}r.\end{aligned} \quad (19)$$

Table 1. Some uncertainties of the Dirac hydrogen-like atom H ($Z = 1$) for $\kappa = -(l+1)$.

(nl)	$(\frac{\langle r\rangle}{a_0})_R$	$(\frac{\langle r\rangle}{a_0})_N$	$(\frac{\Delta r}{a_0})_R$	$(\frac{\Delta r}{a_0})_N$	$(\frac{\Delta r}{\langle r\rangle})_R$	$(\frac{\Delta r}{\langle r\rangle})_N$	$(\frac{\Delta p_r a_0}{\hbar})_R$	$(\frac{a_0 \Delta p_r}{\hbar})_N$	$(\frac{\Delta r \Delta p_r}{\hbar})_R$	$(\frac{\Delta r \Delta p_r}{\hbar})_N$
(10)	1.499 973	1.500 0000	0.866 0177	0.866 0254	0.577 3554	0.577 350 27	1.000 0266	1.000 000 00	0.866 041	0.866 0254
(20)	5.999 883	6.000 000	2.449 468	2.449 4897	0.408 2526	0.408 248 29	0.500 0150	0.500 000 00	1.224 771	1.224 7449
(21)	4.999 973	5.000 000	2.236 062	2.236 0680	0.447 2148	0.447 213 60	0.288 6764	0.288 675 13	0.645 4984	0.645 4972
(30)	13.499 802	13.500 000	4.974 893	4.974 9372	0.368 5160	0.368 513 87	0.333 3413	0.333 333 33	1.658 337	1.658 3124
(31)	12.499 93	12.500 000	4.873 386	4.873 3972	0.389 8732	0.389 871 77	0.248 4535	0.248 452 00	1.210 810	1.210 8053
(32)	10.499 97	10.500 000	3.968 622	3.968 6270	0.377 9650	0.377 964 47	0.149 0715	0.149 071 20	0.591 608	0.591 6080
(40)	23.999 72	24.000 000	8.485 212	8.485 2814	0.353 5546	0.353 553 39	0.250 0049	0.250 000 00	2.121 345	2.121 3203
(41)	22.999 88	23.000 000	8.426 130	8.426 1498	0.366 3553	0.366 354 34	0.204 1253	0.204 124 15	1.719 986	1.719 9806
(42)	20.999 94	21.000 000	7.937 246	7.937 2539	0.377 9651	0.377 964 47	0.158 1143	0.158 113 88	1.254 992	1.254 9900
(43)	17.999 97	18.000 000	5.999 996	6.000 000	0.333 3336	0.333 333 33	0.094 4912	0.094 491 12	0.566 947	0.566 9467
(50)	37.499 64	37.500 000	12.990 286	12.990 381	0.346 4109	0.346 410 16	0.200 0033	0.200 000 00	2.598 100	2.598 0762
(51)	36.499 84	36.500 000	12.951 803	12.951 834	0.354 8454	0.354 844 76	0.171 2706	0.171 269 77	2.218 263	2.218 2576
(52)	34.499 91	34.500 000	12.639 21	12.639 225	0.366 3549	0.366 354 34	0.144 2224	0.144 222 05	1.822 857	1.822 8549
(53)	31.499 95	31.500 000	11.521 71	11.521 719	0.365 7692	0.365 768 85	0.112 1225	0.112 122 38	1.291 843	1.291 8425
(54)	27.499 97	27.500 000	8.291 56	8.291 562	0.301 5115	0.301 511 34	0.066 666 71	0.066 666 67	0.552 771	0.552 7708
(60)	53.999 56	54.000 00	18.493 121	18.493 242	0.342 4680	0.342 467 44	0.166 6690	0.166 666 667	3.082 230	3.082 2070
(61)	52.999 80	53.000 00	18.466 14	18.466 185	0.348 4191	0.348 418 59	0.146 9868	0.146 986 18	2.714 280	2.714 2741
(62)	50.999 88	51.000 00	18.248 27	18.248 288	0.357 8100	0.357 809 56	0.129 0997	0.129 099 44	2.355 847	2.355 8438
(63)	47.999 93	48.000 00	17.492 85	17.492 856	0.364 4348	0.364 434 49	0.109 1091	0.109 108 95	1.908 628	1.908 6270
(64)	43.999 95	44.000 00	15.556 34	15.556 349	0.353 5536	0.353 553 39	0.084 862 58	0.084 862 51	1.320 151	1.320 1509
(65)	38.999 97	39.000 00	10.816 65	10.816 654	0.277 3502	0.277 350 10	0.050 251 91	0.050 251 89	0.543 557	0.543 5573

By substituting equation (4) into equation (19) and then using the recursion relation $zF(1+\alpha, 1+\gamma; z) = \gamma F(1+\alpha, \gamma; z) - \gamma F(\alpha, \gamma; z)$ [15], we can obtain a very complicated expression of $\langle p_r^2 \rangle$, which includes 24 integrals $I_{FF}(n, \Delta n, m, \Delta m, \lambda)$ with different n, Δn, m and Δm. By using package INTEPFFLL [5] and greatly simplifying, we can finally obtain the uncertainty for the radial momentum p_r as follows,

$$\begin{aligned}\Delta p_r &= \sqrt{\langle p_r^2 \rangle - \langle p_r \rangle^2} \\ &= \frac{\hbar Z}{a_0 N^2 \sqrt{2(4\gamma^2 - 1)(N - \kappa)}} \{-4n'^3\gamma(N - 2\kappa) + n'^2[-N(3 + 2\gamma) + 4\kappa(1 + 2\gamma^2)] \\ &\quad - 2n'\gamma(N - 2N^3 + 2N\gamma - 2\kappa + 4N^2\kappa - 2N\kappa^2) + N(1 + 2\gamma)(N - \kappa)^2\}^{1/2}, \qquad (20)\end{aligned}$$

which depends on the quantum numbers n and l and is proportional to the value of charge Z.

Based on the results (17) and (20), we finally obtain a rather complicated expression of the product of the radial position and radial momentum uncertainties,

$$\begin{aligned}\Delta r \Delta p_r = {} & \frac{\hbar}{2N^3\sqrt{2(4\gamma^2 - 1)(N^2 - \kappa)}} \{[n'^2(n'^2 + 2\gamma)^2(2 + n'^2 + 2n'\gamma - \gamma^2) \\ & + n'(n' + 2\gamma)(3 + 4n'(n' + 2\gamma))\kappa^2 - 2N(n' + \gamma)\kappa^3 + (1 + 2n'(n' + 2\gamma))\kappa^4] \\ & \times [-4n'^3\gamma(N - 2\kappa) + n'^2(-N(3 + 2\gamma) + 4\kappa(1 + 2\gamma^2)) - 2n'\gamma(N - 2N^3 + 2N\gamma \\ & - 2\kappa + 4N^2\kappa - 2N\kappa^2) + N(1 + 2\gamma)(N - \kappa)^2]\}^{1/2}, \qquad (21)\end{aligned}$$

from which we find that the product $\Delta r \Delta p_r$ of the radial position–momentum uncertainty relations is exact and analytical. We will study some properties of these uncertainties by taking a few typical values of charge Z in the next section.

3. Some properties of uncertainties in radial position and momentum

Due to the complicated expressions for the uncertainties Δr, Δp_r, relative dispersion $\Delta r/\langle r \rangle$ and product $\Delta r \Delta p_r$, we want to show some interesting features of these uncertainties by doing some typical calculations. Some useful results of $\kappa = -(l+1)$ for $n \in [1, 6]$ and $Z = 1, 11, 37, 87$ with different l are listed in tables 1–4, where we have used the subscripts R

Table 2. Some uncertainties of the Dirac hydrogen-like atom Na ($Z = 11$) for $\kappa = -(l + 1)$.

(nl)	$\left(\frac{\langle r\rangle}{a_0}\right)_R$	$\left(\frac{\langle r\rangle}{a_0}\right)_N$	$\left(\frac{\Delta r}{a_0}\right)_R$	$\left(\frac{\Delta r}{a_0}\right)_N$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_R$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_N$	$\left(\frac{\Delta p_r a_0}{\hbar}\right)_R$	$\left(\frac{a_0 \Delta p_r}{\hbar}\right)_N$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_R$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_N$
(10)	0.136 0701	0.136 363 64	0.078 644 81	0.078 729 58	0.577 9726	0.577 350 27	11.035 688	11.000 000	0.867 900	0.866 0254
(20)	0.544 1710	0.545 4545	0.222 4410	0.222 680 89	0.408 7704	0.408 248 29	5.520 068	5.500 0000	1.227 889	1.224 7449
(21)	0.454 2523	0.454 5455	0.203 2133	0.203 278 91	0.447 3579	0.447 213 60	3.177 134	3.175 426	0.645 6361	0.645 4972
(30)	1.225 0883	1.227 2727	0.451 7800	0.452 267 02	0.368 7734	0.368 513 87	3.677 380	3.666 6667	1.661 367	1.658 3124
(31)	1.135 549	1.136 3636	0.442 9141	0.443 036 11	0.390 0439	0.389 871 77	2.734 932	2.732 9720	1.211 340	1.210 8053
(32)	0.954 252	0.954 5455	0.360 7289	0.360 784 27	0.378 0225	0.377 964 47	1.640 136	1.639 783	0.591 644	0.591 6080
(40)	2.178 746	2.181 8182	0.770 6280	0.771 389 22	0.353 7025	0.353 553 39	2.756 546	2.750 0000	2.124 271	2.121 3203
(41)	2.089 627	2.090 9091	0.765 7918	0.766 013 62	0.366 4730	0.366 354 34	2.246 890	2.245 3656	1.720 650	1.719 9806
(42)	1.908 441	1.909 0909	0.721 4766	0.721 568 54	0.378 0451	0.377 964 47	1.739 758	1.739 2527	1.255 195	1.254 9900
(43)	1.636 071	1.636 3636	0.545 4057	0.545 4545	0.333 3632	0.333 333 33	1.039 522	1.039 4023	0.566 961	0.566 9467
(50)	3.405 135	3.409 0909	1.179 8985	1.180 9437	0.346 5056	0.346 410 16	2.204 392	2.200 0000	2.600 959	2.598 0762
(51)	3.316 446	3.318 1818	1.177 0960	1.177 4394	0.354 9269	0.354 844 76	1.885 120	1.883 9674	2.218 967	2.218 2576
(52)	3.135 391	3.136 3636	1.148 8725	1.149 0204	0.366 4209	0.366 354 34	1.586 916	1.586 4426	1.823 164	1.822 8549
(53)	2.863 071	2.863 6364	1.047 351	1.047 4290	0.365 8137	0.365 768 85	1.233 535	1.233 3462	1.291 944	1.291 8425
(54)	2.499 707	2.500 0000	0.753 734	0.753 7784	0.301 5290	0.301 511 34	0.733 3859	0.733 3333	0.552 778	0.552 7708
(60)	4.904 253	4.909 0909	1.679 8704	1.681 2038	0.342 5334	0.342 467 44	1.836 478	1.833 3333	3.085 045	3.082 2070
(61)	4.816 000	4.818 1818	1.678 2687	1.678 7441	0.348 4777	0.348 418 59	1.617 734	1.616 8480	2.714 993	2.714 2741
(62)	4.635 081	4.636 3636	1.658 716	1.658 9352	0.357 8613	0.357 809 56	1.420 499	1.420 0939	2.356 205	2.355 8438
(63)	4.362 822	4.363 6364	1.590 144	1.590 2596	0.364 4759	0.364 434 49	1.200 395	1.200 1984	1.908 801	1.908 6270
(64)	3.999 486	4.000 000	1.414 143	1.414 2136	0.353 5812	0.353 553 39	0.933 575	0.933 4876	1.320 209	1.320 1509
(65)	3.545 161	3.545 455	0.983 292	0.983 3322	0.277 3616	0.277 350 10	0.552 7978	0.552 7708	0.543 561	0.543 5573

Table 3. Some uncertainties of the Dirac hydrogen-like atom Rb ($Z = 37$) for $\kappa = -(l + 1)$.

(nl)	$\left(\frac{\langle r\rangle}{a_0}\right)_R$	$\left(\frac{\langle r\rangle}{a_0}\right)_N$	$\left(\frac{\Delta r}{a_0}\right)_R$	$\left(\frac{\Delta r}{a_0}\right)_N$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_R$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_N$	$\left(\frac{\Delta p_r a_0}{\hbar}\right)_R$	$\left(\frac{a_0 \Delta p_r}{\hbar}\right)_N$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_R$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_N$
(10)	0.039 536 21	0.040 540 541	0.023 114 35	0.023 406 092	0.584 6374	0.577 350 27	38.456 64	37.000 000	0.888 900	0.866 0254
(20)	0.157 7899	0.162 162 16	0.065 373 42	0.066 202 425	0.414 3068	0.408 248 29	19.316 08	18.500 000	1.262 758	1.224 7449
(21)	0.134 1449	0.135 135 14	0.060 212 45	0.060 434 270	0.448 8611	0.447 213 60	10.746 804	10.680 980	0.647 0914	0.645 4972
(30)	0.357 4131	0.364 864 86	0.132 782 91	0.134 457 76	0.371 5110	0.368 513 87	12.767 51	12.333 333	1.695 307	1.658 3124
(31)	0.335 0914	0.337 837 84	0.131 2995	0.131 713 44	0.391 8319	0.389 871 77	9.268 23	9.192 724	1.216 914	1.210 8053
(32)	0.282 7961	0.283 783 78	0.107 0734	0.107 260 19	0.378 6239	0.377 964 47	5.529 121	5.515 634	0.592 022	0.591 6080
(40)	0.638 1608	0.648 6486	0.226 7187	0.229 331 93	0.355 2690	0.353 553 39	9.514 55	9.250 000	2.157 128	2.121 3203
(41)	0.617 2947	0.621 6216	0.226 9823	0.227 733 78	0.367 7049	0.366 354 34	7.611 316	7.552 593	1.727 634	1.719 9806
(42)	0.565 3777	0.567 5676	0.214 2099	0.214 520 38	0.378 8793	0.377 964 47	5.869 545	5.850 214	1.257 314	1.254 9900
(43)	0.485 4997	0.486 4865	0.161 9976	0.162 162 16	0.333 6719	0.333 333 33	3.500 739	3.496 171	0.567 111	0.566 9467
(50)	1.000 0019	1.013 5135	0.347 5064	0.351 091 38	0.347 5057	0.346 410 16	7.577 15	7.400 000	2.633 108	2.598 0762
(51)	0.980 6302	0.986 4865	0.348 8874	0.350 049 56	0.355 7788	0.354 844 76	6.381 351	6.336 981	2.226 373	2.218 2576
(52)	0.929 1554	0.932 4324	0.341 1014	0.341 600 67	0.367 1091	0.366 354 34	5.354 309	5.336 216	1.826 362	1.822 8549
(53)	0.849 4475	0.851 3514	0.311 1337	0.311 397 80	0.366 2777	0.365 768 85	4.155 741	4.148 528	1.292 991	1.291 8425
(54)	0.742 2569	0.743 2432	0.223 9475	0.224 096 27	0.301 7116	0.301 511 34	2.468 670	2.466 667	0.552 852	0.552 7708
(60)	1.442 9291	1.459 4595	0.495 2463	0.499 817 35	0.343 2229	0.342 467 44	6.293 32	6.166 6667	3.116 742	3.082 2070
(61)	1.425 0670	1.432 4324	0.497 4781	0.499 086 09	0.349 0910	0.348 418 59	5.472 587	5.438 489	2.722 492	2.714 2741
(62)	1.374 060	1.378 3784	0.492 4585	0.493 196 96	0.358 3967	0.357 809 56	4.792 180	4.776 679	2.359 950	2.355 8438
(63)	1.294 557	1.297 2973	0.472 3892	0.472 779 88	0.364 9041	0.364 434 49	4.044 534	4.037 031	1.910 594	1.908 6270
(64)	1.187 461	1.189 1892	0.420 2053	0.420 441 87	0.353 8688	0.353 553 39	3.143 236	3.139 913	1.320 805	1.320 1509
(65)	1.053 068	1.054 0541	0.292 2052	0.292 342 00	0.277 4799	0.277 350 10	1.860 349	1.859 320	0.543 604	0.543 5573

and N to denote the relativistic hydrogen-like atoms case and the non-relativistic hydrogen-like atoms case, respectively. We do not list the results of the Dirac hydrogen-like atoms for other values of charge Z for simplicity. Similarly, we do not list the results of $\kappa = l$ since they are the same as those of $\kappa = -(l + 1)$ for the same n and $l \neq 0$. However, it should be noted that $l \neq 0$ for $\kappa = l$. Moreover, for better visualization some features of these uncertainties are also plotted in figures 1–4. We find that the relativistic corrections to the non-relativistic values of these uncertainties are very small when the value of charge Z is not too big, while the relativistic corrections to them will appear for large Z (e.g., $Z = 87$).

By analysing tables 1–4 and figures 1–4 carefully, we find that there are a few kinds of change rules. First, for the same principal quantum number n, the analytical uncertainties Δp_r and the product $\Delta r \Delta p_r$ decrease as l increases. It should be pointed out that, in the non-relativistic hydrogen-like atoms case, the uncertainties $\langle r\rangle$ and Δr also decrease as l increases

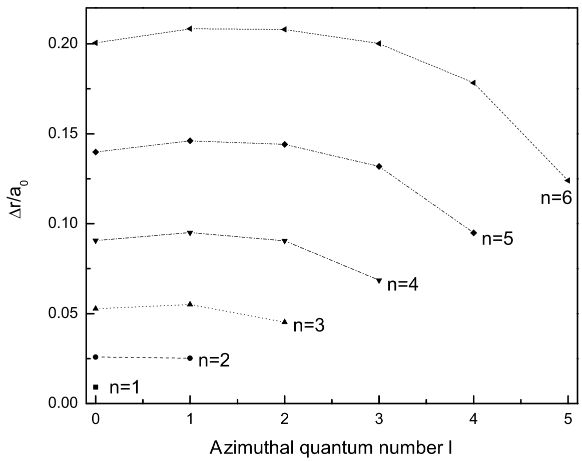

Figure 1. The change rule in the uncertainty $\triangle r$ with respect to the quantum numbers n and l for $Z = 87$. For the same l, this uncertainty increases as n increases.

Table 4. Some uncertainties of the Dirac hydrogen-like atom Fr ($Z = 87$) for $\kappa = -(l+1)$.

(nl)	$\left(\frac{\langle r\rangle}{a_0}\right)_R$	$\left(\frac{\langle r\rangle}{a_0}\right)_N$	$\left(\frac{\Delta r}{a_0}\right)_R$	$\left(\frac{\Delta r}{a_0}\right)_N$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_R$	$\left(\frac{\Delta r}{\langle r\rangle}\right)_N$	$\left(\frac{\Delta p_r a_0}{\hbar}\right)_R$	$\left(\frac{a_0 \Delta p_r}{\hbar}\right)_N$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_R$	$\left(\frac{\Delta r \Delta p_r}{\hbar}\right)_N$
(10)	0.014 626 23	0.017 241 379	0.009 168 359	0.009 954 315	0.626 8435	0.577 350 27	117.851 5	87.000 00	1.080 505	0.866 0254
(20)	0.057 875 43	0.068 965 52	0.025 878 86	0.028 155 055	0.447 1477	0.408 248 29	60.346 6	43.500 000	1.561 700	1.224 7449
(21)	0.055 092 05	0.057 471 26	0.025 164 30	0.025 701 931	0.456 7682	0.447 213 60	26.028 74	25.114 74	0.654 995	0.645 4972
(30)	0.136 105 68	0.155 172 41	0.052 699 01	0.057 183 186	0.387 1918	0.368 513 87	37.785 46	29.000 000	1.991 256	1.658 3124
(31)	0.137 1254	0.143 678 16	0.055 000 15	0.056 016 059	0.401 0938	0.389 871 77	22.661 56	21.615 324	1.246 389	1.210 8053
(32)	0.118 3454	0.120 689 66	0.045 171 22	0.045 616 402	0.381 6895	0.377 964 47	13.149 21	12.969 194	0.593 966	0.591 6080
(40)	0.248 9047	0.275 862 07	0.090 605 22	0.097 531 97	0.364 0157	0.353 553 39	27.024 70	21.750 000	2.448 579	2.121 3203
(41)	0.254 0303	0.264 367 82	0.095 020 85	0.096 852 30	0.374 0532	0.366 354 34	18.571 44	17.758 801	1.764 674	1.719 9806
(42)	0.236 1979	0.241 379 31	0.090 488 07	0.091 232 80	0.383 1028	0.377 964 47	14.013 74	13.755 908	1.268 076	1.254 9900
(43)	0.204 5641	0.206 896 55	0.068 575 68	0.068 965 52	0.335 2283	0.333 333 33	8.280 960	8.220 727	0.567 872	0.566 9467
(50)	0.396 2112	0.431 034 48	0.139 858 02	0.149 314 72	0.352 9886	0.346 410 16	20.895 87	17.400 000	2.922 46	2.598 0762
(51)	0.405 5337	0.419 540 23	0.146 053 01	0.148 871 65	0.360 1501	0.354 844 76	15.513 06	14.900 470	2.265 730	2.218 2576
(52)	0.388 7960	0.396 551 72	0.144 0828	0.145 278 44	0.370 5872	0.366 354 34	12.788 69	12.547 318	1.842 631	1.822 8549
(53)	0.357 5767	0.362 068 97	0.131 8056	0.132 433 55	0.368 6078	0.365 768 85	9.849 72	9.754 647	1.298 248	1.291 8425
(54)	0.313 7649	0.316 091 95	0.094 9538	0.095 305 31	0.302 6274	0.301 511 34	5.826 272	5.800 000	0.553 227	0.552 7708
(60)	0.578 0100	0.620 6897	0.200 541 52	0.212 566 00	0.346 9517	0.342 467 44	16.980 97	14.500 000	3.405 39	3.082 2070
(61)	0.591 5658	0.609 1954	0.208 3653	0.212 255 00	0.352 2267	0.348 418 59	13.257 50	12.787 798	2.762 402	2.714 2741
(62)	0.575 9817	0.586 2069	0.207 9862	0.209 750 43	0.361 0987	0.357 809 56	11.438 36	11.231 652	2.379 022	2.355 8438
(63)	0.545 2586	0.551 7241	0.200 1384	0.201 067 31	0.367 0523	0.364 434 49	9.591 40	9.492 478	1.919 607	1.908 6270
(64)	0.501 6734	0.505 7471	0.178 2485	0.178 808 61	0.355 3078	0.353 553 39	7.426 618	7.383 039	1.323 784	1.320 1509
(65)	0.445 9517	0.448 2759	0.124 0066	0.124 329 35	0.278 0719	0.277 350 10	4.385 371	4.371 914	0.543 815	0.543 5573

except for the uncertainties Δp_r and $\Delta r \Delta p_r$, as discussed by Kuo [4]. On the other hand, for the same n the uncertainty $\langle r\rangle_R$ first increases and then decreases as l increases for large values of charge $Z = 37, 87$, namely, the uncertainty $\langle r\rangle_R$ for states $(n, l = 1)(n = 3-6)$ is biggest. This kind of property does not exist at all in the non-relativistic hydrogen-like atoms case. The difference between them should arise from the relativistic corrections to the non-relativistic values of these uncertainties. Second, all uncertainties for the same n and $l = n - 1$ are smallest in comparison with those for the same n but $l \neq n - 1$. This property is not new since it also exists in the non-relativistic case. Third, it is found that the relativistic corrections to non-relativistic values of these uncertainties are very small when the values of charge Z are not too big, while the relativistic corrections to them will appear when Z is large, in particular for $Z = 87$. Fourth, the product $\Delta r \Delta p_r$ in the non-relativistic case is independent of the value

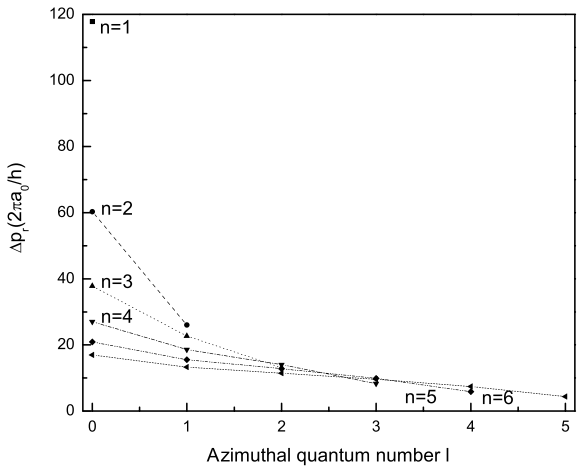

Figure 2. The change rule in the uncertainty $\triangle p_r$ with respect to the quantum numbers n and l for $Z = 87$. For the same n, this uncertainty decreases as l increases. Here $\hbar = h/2\pi$ is used.

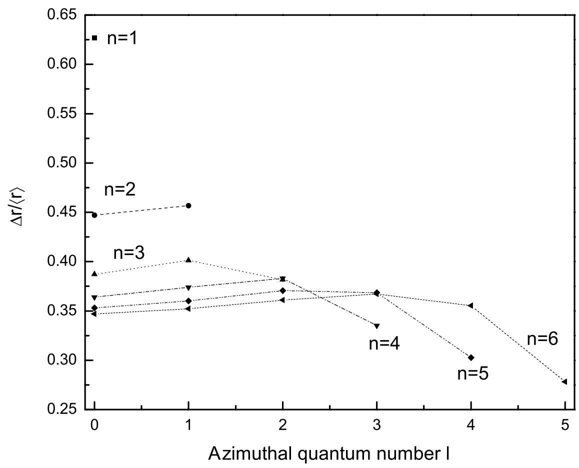

Figure 3. The change rule in the relative dispersion $\triangle r/\langle r\rangle$ with respect to the quantum numbers n and l for $Z = 87$. It is found that there does not exist any explicit change rule for the same n or l.

of charge Z (see equation (37) below), while in the relativistic case it is related with the value of charge Z, in particular when Z is very large, say $Z = 87$, the relativistic correction to it appears. Furthermore, $(\Delta p_r)_R$ and $(\Delta r \Delta p_r)_R$ are almost equal to those in the non-relativistic case when n is large. That is to say, the relativistic corrections to the non-relativistic values become very small and can be ignored when one analyses the behaviour of the uncertainties of the radius r and radial momentum p_r or their product for large n. Fifth, for the same l both Δr and $\Delta r \Delta p_r$ increase as n increases, while both Δp_r and $\Delta r/\langle r\rangle$ decrease. This property exists both in the non-relativistic case and in the relativistic case.

4. Special cases for $l = 0$ and $l = n - 1$

In order to have an insight into the features of those uncertainties, we attempt to study two special cases for $l = 0$ and $l = n - 1$, as shown in [4]. First, let us consider the special

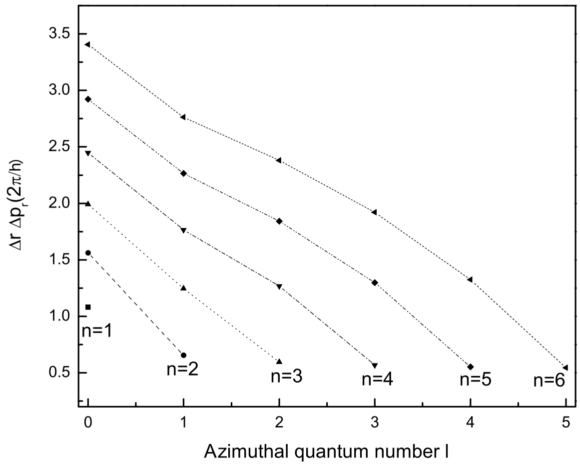

Figure 4. The change rule in the uncertainty $\triangle r \triangle p_r$ with respect to the quantum numbers n and l for $Z = 87$. For the same l, this uncertainty increases as n increases, while for the same n, it decreases as l increases.

case $l = 0$. For this case, we have $\kappa = -(l+1) = -1$. The corresponding results for $\kappa = -(l+1) = -1$ are given by

$$\Delta r|_{\kappa=-1} = \frac{\omega}{2Z\zeta}, \qquad \Delta p_r|_{\kappa=-1} = \frac{\tau}{a_0\zeta^2\sqrt{(4\gamma^2-1)(1+\zeta)}}, \tag{22}$$

$$\Delta r\,\Delta p_r|_{\kappa=-1} = \frac{\tau\omega}{2Za_0\zeta^3\sqrt{(4\gamma^2-1)(1+\zeta)}}, \tag{23}$$

$$\left.\frac{\Delta r}{\langle r\rangle}\right|_{\kappa=-1} = \frac{\omega}{a_0[-5+3n^3+\zeta+9n^2(\gamma-1)+11\gamma-6\gamma^2+n(11-18\gamma+6\gamma^2)]}, \tag{24}$$

where

$$\zeta = \sqrt{2-2\gamma+n(-2+n+2\gamma)}, \tag{25}$$

$$\omega = a_0\{1+(n-1)(n+2\gamma-1)[5+4(n-1)(n+2\gamma-1)]+2(n+\gamma-1)\zeta + (n-1)^2(n+2\gamma-1)^2[n^2+2n(\gamma-1)-(\gamma-1)(3+\gamma)]\}^{1/2}, \tag{26}$$

$$\tau = \hbar Z\{n[2-n+2n\gamma+4(n-1)\gamma^2]+[n(2-n)-2\gamma+4n\gamma+4(n-1)^2\gamma^2]\zeta\}^{1/2}. \tag{27}$$

Second, let us study the special case $l = n-1$. Here, we study two different cases for both $\kappa = -(l+1) = -n$ and $\kappa = l = n-1$. For $\kappa = -(l+1) = -n$, we have

$$\Delta r|_{\kappa=-n} = \frac{na_0\sqrt{1+2\gamma}}{2Z}, \qquad \Delta p_r|_{\kappa=-n} = \frac{\hbar Z}{na_0\sqrt{2\gamma-1}}, \tag{28}$$

$$\frac{\Delta r}{\langle r\rangle}|_{\kappa=-n} = \frac{1}{\sqrt{1+2\gamma}}, \qquad \Delta r\,\Delta p_r|_{\kappa=-n} = \frac{\hbar}{2\sqrt{\frac{2\gamma-1}{2\gamma+1}}}. \tag{29}$$

For $\kappa = l = n-1$, however, we can obtain the following complicated results:

$$\Delta r|_{\kappa=n-1} = \frac{a_0\varsigma}{2Z\zeta'}, \qquad \Delta p_r|_{\kappa=n-1} = \frac{\hbar Z\nu}{a_0\zeta'^2\sqrt{(4\gamma^2-1)(1-n+\zeta')}}, \tag{30}$$

$$\Delta r\,\Delta p_r|_{\kappa=n-1} = \frac{\hbar\varsigma\nu}{2\zeta'^3\sqrt{(4\gamma^2-1)(1-n+\zeta')}}, \tag{31}$$

$$\left.\frac{\Delta r}{\langle r\rangle}\right|_{\kappa=n-1} = \frac{\varsigma}{(1-n)\zeta' + (1+\gamma)[5+2n(n-2)+6\gamma]}, \tag{32}$$

where

$$\zeta' = \sqrt{2-2n+n^2+2\gamma}, \tag{33}$$

$$\varsigma = [(\gamma-3)(1+\gamma)(1+2\gamma)^2 + 2(n-1)^3(1+\gamma)\zeta' + (n-1)^4(3+4\gamma) + (n-1)^2(1+2\gamma)(7+8\gamma)]^{1/2}, \tag{34}$$

$$\nu = \{\zeta'[n(n-2)+6(n-1)^2\gamma+4\gamma^2] - (n-1)[n(n-2)+2[4+3n(n-2)]\gamma+8\gamma^2]\}^{1/2}. \tag{35}$$

5. Non-relativistic limit

We now study the non-relativistic limit of the Dirac hydrogen-like atoms. According to [1], we have $\kappa = \gamma = l$, $N = n$ and $\epsilon = 1$ if ignoring α in comparison with unity. Thus, $g(r)$ vanishes because of the factor $1-\epsilon$, and $f(r)$ is nothing but the normalized Schrödinger wavefunction if replacing κ by l. Similarly, in this limit we are able to obtain the following uncertainties of the non-relativistic hydrogen-like atoms,

$$\Delta r = \frac{a_0}{2Z}\sqrt{n^2(2+n^2) - l^2(l+1)^2}, \qquad \Delta p_r = \frac{Z\hbar}{na_0}\sqrt{1-\frac{2l(l+1)}{n(1+2l)}}, \tag{36}$$

$$\frac{\Delta r}{\langle r\rangle} = \frac{\sqrt{n^2(2+n)^2 - l^2(l+1)^2}}{3n^2 - l(l+1)}, \qquad \Delta r\,\Delta p_r = \frac{\hbar}{2}\sqrt{1-\frac{2l(l+1)}{n(1+2l)}}\sqrt{(2+n^2) - \frac{l^2(l+1)^2}{n^2}}. \tag{37}$$

Similarly, we have for $\kappa = -(l+1)$ or l and $l = n-1$

$$\Delta r = \frac{a_0 n\sqrt{1+2n}}{2Z}, \qquad \Delta p_r = \frac{\hbar Z}{a_0 n\sqrt{2n-1}},$$
$$\frac{\Delta r}{\langle r\rangle} = \frac{1}{\sqrt{1+2n}}, \qquad \Delta r\,\Delta p_r = \frac{\hbar}{2}\sqrt{\frac{2n+1}{2n-1}}. \tag{38}$$

For the $l = 0$ case, however, we have

$$\Delta r = \frac{na_0}{2Z}\sqrt{2+n^2}, \qquad \Delta p_r = \frac{\hbar Z}{na_0},$$
$$\frac{\Delta r}{\langle r\rangle} = \frac{\sqrt{2+n^2}}{3n}, \qquad \Delta r\,\Delta p_r = \frac{\hbar}{2}\sqrt{n^2+2}. \tag{39}$$

6. Concluding remarks

With the aid of the MATHEMATICA package INTEPFFLL, we have shown that the uncertainties Δr, Δp_r, the relative dispersion $\Delta r/\langle r\rangle$ and the product $\Delta r \Delta p_r$ for the Dirac hydrogen-like atoms can be obtained analytically. All these quantities are found to depend on the quantum numbers n and l as well as the value of charge Z. However, it should be pointed out that $\Delta r \Delta p_r$ in the non-relativistic case is independent of the value of charge Z. On the other hand, we have found that the relativistic corrections to the non-relativistic values of these uncertainties are very small when the value of charge Z is not too big, while the relativistic corrections to them will appear for large Z (e.g., $Z = 87$). In addition, we have also found that the uncertainty $\langle r\rangle_R$ for states $(n, l = 1)(n \in [3, 6])$ is biggest. This property does not exist at all in the non-relativistic case. It should be mentioned that these properties are new in comparison with those in the non-relativistic case. We do not want to summarize other properties of these uncertainties for simplicity since they have been given in section 3.

The fact that the uncertainties Δr and Δp_r presented in this work are found to be exact and analytical suggests that both the position and momentum of an electron in the relativistic Dirac hydrogen-like atoms can be measured simultaneously with known uncertainties if the wavefunctions of that electron could be specified. A similar conclusion was also drawn by Kuo in the non-relativistic case [4], in which he also explained the inequality '$\geqslant$' appearing in the uncertainty relation of Heisenberg. The detailed information can be found in [4]. Before ending this work, we make a remark here. The present case tells us that because the wavefunctions of this quantum system are put into the calculations of the position–momentum uncertainties, we obtain the exact uncertainties. Therefore, the present approach may be used to investigate these uncertainties for other quantum systems if the wavefunctions are known exactly.

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